Differentiation of measures

We say that a measure μ is locally finite if for every point x we can find $r>0$ s.t. $\mu(B(x,r))<\infty$, where $B(x,r)$ is the closed ball centered at x with radius r.

Definition Let μ, λ be locally finite Borel measures on $\mathbb{R}^d$. The upper and lower derivatives of μ with respect to λ at $x \in \mathbb{R}^d$ are

w.r.t.

$\frac{0}{0} := \infty$

$$\overline{D}(\mu,\lambda,x) := \overline{\lim_{r\downarrow 0}} \frac{\mu(B(x,r))}{\lambda(B(x,r))} ; \underline{D}(\mu,\lambda,x) = \underline{\lim_{r\downarrow 0}} \frac{\mu(B(x,r))}{\lambda(B(x,r))}$$

At the points x where the limit above exists we write $D(\mu,\lambda,x) := \lim_{r\downarrow 0} \frac{\mu(B(x,r))}{\lambda(B(x,r))}$, and we say that $D(\mu,\lambda,x)$ is the derivative of μ w.r.t. λ.

Homework (neither collected nor compulsory)

Prove that the functions $x \mapsto \overline{D}(\mu,\lambda,x)$, $x \mapsto \underline{D}(\mu,\lambda,x)$ are Borel measurable functions, in the special case when $\lambda = \mathcal{L}^d$ (d-dimensional Lebesgue measure), via the following steps:

(a) Prove that $x \mapsto \mu(B(x,r))$ is upper semmi continuous. That is whenever $x_i \to x$ then

$$\lim_{n\to\infty} \mu(B(x_i, r)) \le \mu(B(x,r)).$$

(b) Prove that in the definitions of $\overline{D}(\mu,\lambda,x)$ & $\underline{D}(\mu,\lambda,x)$ we can restrict ourselves to positive rational r.

To see this you can use that

- $r \mapsto \mu(B(x,r))$ is monotone increasing in r,
- $r \mapsto \mathcal{L}^d(B(x,r))$ is continuous in r.

(c) Use that the infima & suprema of Borel functions are Borel functions.

Recall: We say that μ is absolute continuous w.r.t. λ if $\boxed{\lambda(A)=0 \Rightarrow \mu(A)=0}$.

Theorem 1 (Differentiation) Let μ & λ be Radon measures.

(1) For λ-a.a. $x \in \mathbb{R}^d$, $D(\mu,\lambda,x)$ exists & finite.

(2) $\int_B D(\mu,\lambda,x)\,d\lambda(x) \le \mu(B)$ for any $B \subset \mathbb{R}^d$ Borel set, with equality if $\mu \ll \lambda$.

(3) $\mu \ll \lambda \iff \underline{D}(\mu,\lambda,x) < \infty$ for μ-a.a. $x \in \mathbb{R}^d$.

In order to prove this theorem we need a lemma.

Lemma 2 Let μ, λ be Radon measures on $\mathbb{R}^d$, $0 < t < \infty$ & $A \subset \mathbb{R}^d$ such that

(1) $\left(\forall x \in A,\ \underline{D}(\mu, \lambda, x) \le t\right) \Rightarrow \mu(A) \le t \cdot \lambda(A)$,

(2) $\left(\forall x \in A,\ \overline{D}(\mu, \lambda, x) \ge t\right) \Rightarrow \mu(A) \ge t \cdot \lambda(A)$.

Proof of the Lemma part (1) Let $\varepsilon > 0$. Choose an open set U s.t. $A \subset U$ & $\lambda(U) < \lambda(A) + \varepsilon$.

Now we apply the Vitali Covering Theorem for Radon measures as follows: We define a collection of balls: $\forall x \in A$ pick a closed ball B_x centered at x s.t.

$$\text{(1)} \quad \boxed{B_x \subset U, \quad \mu(B_x) < (t+\varepsilon)\lambda(B_x)}$$

Let $\mathcal{B} := \{B_x\}_{x \in A}$. Now we apply the Vitali Covering Theorem for Radon measures for $\mathcal{B}$, A and μ. This yields the existence of disjoint closed balls $\{B_i\}_{i=1}^{\infty}$ s.t.

$$B_i \subset U; \quad \mu(B_i) < (t+\varepsilon)\lambda(B_i); \quad \mu\Big(A \setminus \bigcup_{i=1}^{\infty} B_i\Big) = 0.$$

Then

$$\mu(A) \le \sum_i \mu(B_i) \le (t+\varepsilon) \sum_i \lambda(B_i) \le (t+\varepsilon)\lambda(U) \le (t+\varepsilon)(\lambda(A)+\varepsilon)$$

disjoint balls

Let $\varepsilon \searrow 0$ to get $\mu(A) \le t \cdot \lambda(A)$. This completes the proof of part (1). The proof of part (2) is similar and left as homework. ∎ Now we prove Thm 1.

Proof of Theorem 1

For $0<r<\infty$ & $0<s<t<\infty$ let

$$A_{s,t,r} := \{x \in B(0,r) : \underline{D}(\mu,\lambda,x) \leq s < t \leq \overline{D}(\mu,\lambda,x)\},$$

$$A_{u,r} := \{x \in B(0,r) : \overline{D}(\mu,\lambda,x) \geq u\}.$$

Lemma 2 yields that

$$t\,\lambda(A_{s,t,r}) \leq \mu(A_{s,t,r}) \leq s\cdot\lambda(A_{s,t,r}) < \infty \quad (*)$$

$$u\cdot\lambda(A_{u,r}) \leq \mu(A_{u,r}) \leq \mu(B(0,r)) < \infty. \quad (**)$$

$(*) \Rightarrow \lambda(A_{s,t,r}) = 0$ since $s<t$.

$(**) \Rightarrow \lambda\left(\bigcap_{u>0} A_{u,r}\right) = \lim_{u\to\infty} \lambda(A_{u,r}) = 0.$

$G := \{x : D(\mu,\lambda,x) \text{ exists } \& \ D(\mu,\lambda,x) < \infty\}.$

Then

$$G^c = \left(\bigcup_{\substack{s,t\in\mathbb{Q}^+\\ r\in\mathbb{N}^+}} A_{s,t,r}\right) \cup \left(\bigcap_{u>0} A_{u,r}\right)$$

The λ measure of any of these sets is zero.

The λ measure of this is also zero.

Hence $\lambda(G^c)=0$. This completes the proof of part (a).

Proof of Part (2) Let $1 < t < \infty$, $B \subset \mathbb{R}^d$ Borel set.

$$B_p := \{x \in B : t^p \le D(\mu,\lambda,x) < t^{p+1}\} \quad p \in \mathbb{Z}.$$

Below we use part (1) of this Thm. & part (2) of Lemma 2:

$$\int_B D(\mu,\lambda,x)\, d\lambda(x) = \sum_{p=-\infty}^{\infty} \int_{B_p} D(\mu,\lambda,x)\, d\lambda(x) \le \sum_{p=-\infty}^{\infty} t^{p+1} \cdot \lambda(B_p)$$

$$\le t \cdot \sum_{p=-\infty}^{\infty} \mu(B_p) = t\mu(B)$$

$x \in B_p \Rightarrow \overline{D}(\mu,\lambda,x) > t^p \Rightarrow \mu(B_p) \ge t^p \lambda(B_p)$. Now let $t \downarrow 1$

$$\int_B D(\mu,\lambda,x)\, d\lambda(x) \le \mu(B)$$

If $\mu \ll \lambda$ then any λ-measure zero set is also μ-measure zero set. Hence

$$D(\mu,\lambda,x) = D(\lambda,\mu,x)^{-1} > 0 \text{ for } \mu\text{-a.a. } x.$$

Then $\mu(B) = \sum_{p=-\infty}^{\infty} \mu(B_p)$

$$\int_B D(\mu,\lambda,x)\, d(\lambda) = \sum_{p=-\infty}^{\infty} \int_{B_p} D(\mu,\lambda,x)\, d\lambda(x) \ge \sum_{p=-\infty}^{\infty} t^p \cdot \lambda(B_p) \ge \sum_{p=-\infty}^{\infty} \mu(B_p)$$

$$= \mu(B).$$

Hence $\int_B D(\mu,\lambda,x)\, d\lambda(x) = \mu(B)$.

Proof of Part (3) Assume that $\underline{D}(\mu,\lambda,x)<\infty$ for μ-a.a $x\in\mathbb{R}^d$.

Let $A\subset\mathbb{R}^d$ with $\lambda(A)=0$. We want to prove that $\mu(A)=0$.

We write $A_u := \{x\in A : \underline{D}(\mu,\lambda,x)\le u\}$. Then

By part (1) of Lemma 2 we have $\mu(A_u)\le u\cdot\lambda(A)$.

Hence $\mu(A_u)\le u\cdot\lambda(A)=0$. By our assumption:

$\mu(\mathbb{R}^d\setminus\{x : \underline{D}(\mu,\lambda,x)<\infty\})=0$. Hence

$\mu(\mathbb{R}^d\setminus\bigcup_{u=1}^{\infty} A_u)=0$. On the other hand,

we have seen that $\mu(A_u)=0$. That is

$\mu(A)=0$. In this way we have verified that

$\lambda(A)=0 \Longrightarrow \mu(A)=0$. That is $\mu\ll\lambda$.

This completes the proof of the differentiation of measures theorem. ∎